Rutgers University: Complex Variables and Advanced Calculus Written Qualifying Exam

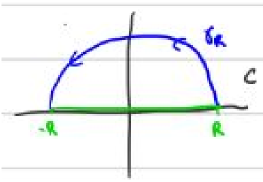
January 2011 Day 1: Problem 2 Solution

Exercise. Use contour integration

$$\int_0^\infty \frac{1}{(1+x^2)^2} dx$$

Solution.

$$\int_0^\infty \frac{1}{(1+x^2)^2} dx = \int_0^\infty \frac{1}{(x-i)^2(x+i)^2} dx$$
$$= \frac{1}{2}\int_{-\infty}^\infty \frac{1}{(x-i)^2(x+i)^2} dx$$
$$= \frac{1}{2}\lim_{R\to\infty}\int_{-R}^R \frac{1}{(x-i)^2(x+i)^2} dx \qquad \text{poles } \pm i \text{ with degree 2}$$



$$\int_C f(z)dz = \int_{-R}^R f(x)dx + \int_{\gamma_R} f(x)dx$$
$$\implies \int_{-R}^R f(x)dx = \int_C f(z)dz - \int_{\gamma_R} f(x)dx$$
$$I = \frac{1}{2}\lim_{R\to\infty}\int_{-R}^R \frac{1}{(x-i)^2(x+i)^2} dx = \frac{1}{2}\lim_{R\to\infty}\left(\int_C f(z)dz - \int_{\gamma_R} f(x)dx\right)$$

$\int_C f(z)dz$: for $R>1, i$ is the only pole in C

$$\int_C f(z)dz = 2\pi i Res(f,i) = 2\pi i \lim_{z\to i}\frac{d}{dz}\left(\frac{1}{(z+i)^2}\right)$$
$$= 2\pi i \lim_{z\to i}\frac{-2}{(z+i)^3} = \frac{-4\pi i}{(2i)^3} = \frac{\pi}{2}$$

$$\int_{\gamma_R} f(x)dx : \left|\int_{\gamma_R} f(x)dx\right| \le \int_{\gamma_R}|f(x)|dx \le \underbrace{\ell(\gamma_R)}_{=\pi R}\cdot\max|f|$$

$$|f| = \left|\frac{1}{(1+x^2)^2}\right| \qquad |(1+z^2)|^2 = |1+z^2|\cdot|1+z^2|$$
$$> (|z^2|-1)(|z^2|-1) = (R^2-1)^2$$
$$\left|\frac{1}{1+z^2}\right| < \frac{1}{(R^2-1)^2}$$
$$\implies \left|\lim_{R\to\infty}\int_{\gamma_R} f(x)dx\right| < \lim_{R\to\infty}\pi R\frac{1}{(R^2-1)^2} = 0$$
$$\implies \lim_{R\to\infty}\int_{\gamma_R} f(x)dx = 0$$
$$I = \frac{1}{2}\lim_{R\to\infty}\left(\int_C f(z)dz - \int_{\gamma_R} f(x)dx\right) = \frac{1}{2}\left(\frac{\pi}{2}-0\right) = \boxed{\frac{\pi}{4}}$$